

Rotating black hole orbit functionals in the frequency domain

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In many astrophysical problems, it is important to understand the behavior of functions that come from rotating (Kerr) black hole orbits. It can be particularly useful to work with the frequency domain representation of those functions, in order to bring out their harmonic dependence upon the fundamental orbital frequencies of Kerr black holes. Although, as has recently been shown by W. Schmidt, such a frequency domain representation must exist, the coupled nature of a black hole orbit's r and θ motions makes it difficult to construct such a representation in practice. Combining Schmidt's description with a clever choice of timelike coordinate suggested by Y. Mino, we have developed a simple procedure that sidesteps this difficulty. One first Fourier expands all quantities using Mino's time parameter λ . In particular, the observer's time t is decomposed with λ . The frequency domain description is then built from the λ -Fourier expansion and the expansion of t . We have found this procedure to be quite simple to implement, and to be applicable to a wide class of functionals. We test the procedure using a simple test function, and then apply it to a particularly interesting case, the Weyl curvature scalar ψ_4 used in black hole perturbation theory.

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I. INTRODUCTION

The black holes which appear to exist in a wide range of masses throughout the universe (see, e.g. Refs. [1, 2, 3, 4, 5, 6, 7, 8]) are most likely described by the Kerr solution of general relativity. The charged generalization is unlikely to be interesting, as macroscopic charged objects should be rapidly neutralized by astrophysical plasma. The Schwarzschild limit is an unrealistic idealization given how unlikely it is for an astrophysical macroscopic object to have precisely zero spin. This motivates a need to thoroughly understand phenomena in the vicinity of Kerr black holes. Such an understanding becomes quite important as studies probe ever more deeply into black holes' strong fields.

Of particular interest to many applications is an understanding of Kerr black hole orbits. In the language of general relativity, "orbits" are bound, stable geodesic trajectories. It is a relatively simple matter to write down the equations governing these orbits and to integrate in the time domain to find the detailed trajectory that a body will follow.

These orbits have a rich phenomenology, owing to the complicated shape of the hole's gravitational "potential". At largish radii ($r \gtrsim 20$ times the radius of the hole), a generic orbit is not too different from the ellipses of Newtonian theory. However, the plane in which this ellipse lies precesses (due largely to the spin of the black hole and the oblateness of the hole's geometry), and the ellipse precesses within that precessing plane. We can identify two fundamental orbital frequencies: a frequency Ω_r characterizing the radial motion (from periapsis to apoapsis and back), and a frequency Ω_θ characterizing the latitudinal motion. A third frequency of somewhat different nature describes the average secular accumulation of the angle about the hole's symmetry axis, and is denoted by Ω_ϕ . The various precessions of the orbit are due to mismatches between these frequencies: the orbital plane precesses at $\Omega_\phi - \Omega_\theta$; the orbital ellipse precesses at $\Omega_\phi - \Omega_r$. Closed form expressions for all three of these frequencies have recently been worked out by W. Schmidt [9]. In the deep strong field of the hole, the frequencies become so different that the qualitative picture given above — a precessing ellipse on a precessing plane — ceases to be useful. The orbits just become complicated and messy.

Despite this complicated nature, a wide class of functions of black hole orbits are completely described by the frequencies Ω_r and Ω_θ . Any function of the form $f[r(t), \theta(t)]$ (a common functional form for black hole orbits, since

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the metric is independent of both t and ϕ) can be expanded as

$$f[r(t), \theta(t)] = \sum_{kn} f_{kn} e^{-ik\Omega_\theta t} e^{-in\Omega_r t} . \quad (1.1)$$

Unless otherwise noted, the index of all sums runs from $-\infty$ to ∞ . The fact that such expansions exist is very useful, since it suggests we can Fourier analyze a wide class of interesting orbit functionals to understand their harmonic dependence upon the orbital frequencies.

Some functions have a more complicated form depending on all four components of the orbital worldline, $z^\alpha = (t, r, \theta, \phi)$. A similar, but slightly modified, expansion can be written down which handles functions of this sort. Such an expansion is needed, for example, to give the harmonic decomposition of an orbiting body's stress-energy tensor, used in frequency domain perturbation theory of Kerr black holes [10]. One could also imagine using this harmonic expansion to describe the emission spectrum of hot material accreting onto a black hole. This could facilitate identifying features that are imprinted upon a black hole's x-ray spectrum.

Actually computing the expansion coefficients f_{kn} turns out to be somewhat difficult. This is fundamentally because the r and θ motions of a black hole orbit are coupled, and as a result are not periodic in coordinate time t (or proper time τ). This difficulty can be fixed by working with a time variable λ , recently suggested by Y. Mino [13], which decouples the r and θ motions. With respect to λ , the r and θ motions are truly periodic. In contrast to the time domain expansion (1.1), a similar expansion using λ is straightforward to compute.

The clocks of distant observers tick at evenly spaced intervals of t , not λ . For the purpose of describing quantities that could be measured by such observers, the t expansion is more useful than the λ expansion. Fortunately, it is straightforward to convert. That is the subject of this paper. The key observation is that observer time t contains oscillatory elements that are periodic with respect to Mino's time λ . Thus, t itself can be expanded in a Fourier series of λ -frequency harmonics.

The remainder of this paper describes our prescription. In Sec. II, we briefly discuss the t -domain description of the orbits. We then show how Mino's time λ fixes many of the difficulties associated with these orbits in Sec. III. In Sec. IV, we show how to use a λ expansion to compute the t expansion coefficients f_{kn} . In Sec. V, we apply this technique first to a relatively simple function of black hole orbits, and then to the Weyl curvature scalar ψ_4 , demonstrating that everything works quite robustly. Appendix A discusses some important details related to implementation of these techniques.

II. ORBITS IN BOYER-LINDQUIST TIME

The geodesic equations that govern Kerr black hole orbits are usually presented in the following ‘‘classic’’ form [14]:

$$\rho^4 \left(\frac{dr}{d\tau} \right)^2 = [E(r^2 + a^2) - aL_z]^2 - \Delta [r^2 + (L_z - aE)^2 + Q] \equiv R(r) , \quad (2.1)$$

$$\rho^4 \left(\frac{d\theta}{d\tau} \right)^2 = Q - \cot^2 \theta L_z^2 - a^2 \cos^2 \theta (1 - E^2) \equiv \Theta(\theta) , \quad (2.2)$$

$$\rho^2 \left(\frac{d\phi}{d\tau} \right) = \csc^2 \theta L_z + aE \left(\frac{r^2 + a^2}{\Delta} - 1 \right) - \frac{a^2 L_z}{\Delta} \equiv \Phi(r, \theta) , \quad (2.3)$$

$$\rho^2 \left(\frac{dt}{d\tau} \right) = E \left[\frac{(r^2 + a^2)^2}{\Delta} - a^2 \sin^2 \theta \right] + aL_z \left(1 - \frac{r^2 + a^2}{\Delta} \right) \equiv T(r, \theta) . \quad (2.4)$$

Up to initial conditions, orbits are specified by the quantities E , L_z , and Q (‘‘energy’’, ‘‘z-component of angular momentum’’, and ‘‘Carter constant’’); these quantities are conserved along any orbit of the family. For notational simplicity, we have put $\rho^2 = r^2 + a^2 \cos^2 \theta$ and $\Delta = r^2 - 2Mr + a^2$. Note that Eqs. (2.1) and (2.2) have been divided by μ^2 , and Eqs. (2.3) and (2.4) by μ (where μ is the mass of a small body in an orbit); E , L_z , and Q are thus the specific energy, angular momentum and Carter constant. The parameter τ is proper time measured along the orbit; t is Boyer-Lindquist coordinate time. We choose $0 \leq a \leq M$; prograde and retrograde orbits are distinguished by an orbital inclination angle rather than the sign of the hole's spin.

By picking initial conditions and physically reasonable values of the constants E , L_z , and Q , one can integrate these equations to obtain a worldline parameterized by proper time τ along the orbit. Schmidt [9] has derived formulae for these constants as functions of an orbit's semi-latus rectum p , eccentricity e , and an inclination angle ι ; further discussion of these parameters is given in Appendix A. Schmidt's formulae do not work well for circular orbits ($e = 0$). Formulae which apply to that case were originally worked out by Shakura [15]; we use a parameterization which was originally derived by Williams [16], and then re-derived by Hughes [17].

For the purpose of understanding quantities which could be measured by distant observers, proper time is not a particularly good choice of parameterization for the orbit — it is connected to the orbit itself, and so contains components which oscillate with respect to the clocks of distant observers. Since the Boyer-Lindquist time coordinate t reduces at large radius to time as measured by distant observers, one should parameterize with t rather than τ . It is trivial to convert: just divide the geodesic equations in τ by $dt/d\tau$ to obtain equations in t :

$$\frac{dr}{dt} = \frac{dr}{d\tau} \left(\frac{dt}{d\tau} \right)^{-1}, \quad (2.5)$$

and likewise for $d\theta/dt$ and $d\phi/dt$. Then, pick initial conditions and an allowed set of orbital constants (E, L_z, Q), and integrate to find $\mathbf{z}(t) = [r(t), \theta(t), \phi(t)]$.

Using elegant Hamilton-Jacobi techniques, W. Schmidt [9] has recently shown that bound orbits satisfying these equations are characterized by multiply-periodic motion in r , θ , and ϕ . These motions are given by three fundamental frequencies, Ω_r , Ω_θ , and Ω_ϕ . In fact, the frequency Ω_ϕ can be considered less fundamental than Ω_r and Ω_θ . This is because the ϕ orbital motion corresponds (in the language of Goldstein [11]) to a *rotation*-type periodic motion, rather than an oscillatory or *libration*-type periodicity. The frequency Ω_ϕ is the average rate at which ϕ accumulates over an orbit. Because $d\phi/dt$ depends only on r and θ , deviations from that average accumulation are oscillations at the r and θ frequencies:

$$\phi(t) = \Omega_\phi t + \sum_{kn} \varphi_{kn} e^{-ik\Omega_\theta t} e^{-in\Omega_r t}. \quad (2.6)$$

Physically, one can imagine analyzing black hole orbits in a frame that co-rotates at the frequency Ω_ϕ . In that corotating frame, the rotation-type periodicity at Ω_ϕ is removed, and only the libration-type oscillations at harmonics of Ω_r and Ω_θ remain (see also discussion in Ref. [11], pp. 466 – 467).

By this logic, many functions $f[\mathbf{z}(t)]$ can be reduced to functions of r and θ only. It is then possible to expand in a Fourier series as

$$f[r(t), \theta(t)] = \sum_{kn} f_{kn} e^{-i(k\Omega_\theta + n\Omega_r)t}. \quad (2.7)$$

Unfortunately, the functions $r(t)$ and $\theta(t)$ are in general not periodic (although they are in the Newtonian limit where all the orbital frequencies are identical). This not-quite-periodic character is fundamentally due to the coupling the r and θ motions in Eqs. (2.1) and (2.2): the functions $(\rho^2 dt/d\tau)^{-2} R$ and $(\rho^2 dt/d\tau)^{-2} \Theta$ each depend explicitly on both r and θ . (Note that this coupling remains if we use proper time along the orbit τ as our parameterization.) The non-separated nature of the r and θ motions makes it difficult to compute the coefficients f_{kn} appearing in Eq. (2.7). If the motions separated, one could define angle variables $w^r \equiv \Omega_r t$ and $w^\theta \equiv \Omega_\theta t$, such that r would be a function only of w^r and θ a function only of w^θ [11, 12]. Computing the coefficients f_{kn} would then be straightforward (see, e.g., Ref. [11], p. 466). Since the motions do not in fact separate, the angles w^r and w^θ are not well defined. An alternative scheme to compute the Fourier series coefficients appears necessary.

III. ORBITS IN MINO TIME

In a recent paper, Y. Mino [13] introduced a new parameterization of Kerr geodesic motion which separates the r and θ motion. In terms of what we shall call “Mino time” λ , the geodesic equations become

$$\left(\frac{dr}{d\lambda} \right)^2 = R(r), \quad (3.1)$$

$$\left(\frac{d\theta}{d\lambda} \right)^2 = \Theta(\theta), \quad (3.2)$$

$$\frac{d\phi}{d\lambda} = \Phi(r, \theta), \quad (3.3)$$

$$\frac{dt}{d\lambda} = T(r, \theta), \quad (3.4)$$

where $R(r)$, $\Theta(\theta)$, $\Phi(r, \theta)$, and $T(r, \theta)$ are defined in Eqs. (2.1)–(2.4). The r and θ motions are now strictly periodic functions:

$$\begin{aligned} r(\lambda) &= r(\lambda + n\Lambda_r), \\ \theta(\lambda) &= \theta(\lambda + n\Lambda_\theta), \end{aligned} \quad (3.5)$$

where n is any integer and the periods are given by

$$\Lambda_r = 2 \int_{r_{\text{peri}}}^{r_{\text{ap}}} \frac{dr}{R(r)^{1/2}}, \quad (3.6)$$

$$\Lambda_\theta = 4 \int_{\theta_{\text{min}}}^{\pi/2} \frac{d\theta}{\Theta(\theta)^{1/2}}. \quad (3.7)$$

The radial motion is taken to range between periapsis, r_{peri} , and apoapsis, r_{ap} ; the θ motion ranges from a minimum θ_{min} to a maximum $\pi - \theta_{\text{min}}$. (With a particular reparameterization, we can write the Λ_r integral in such a way that it behaves well as we approach the limit of circular orbits, $r_{\text{peri}} \rightarrow r_{\text{ap}}$. Likewise it is simple to reparameterize such that Λ_θ is well behaved in the equatorial orbit limit, $\theta_{\text{min}} \rightarrow \pi/2$. See Appendix A.)

For what follows, it will be useful to define the following frequencies conjugate to λ :

$$\Upsilon_{r,\theta} = 2\pi/\Lambda_{r,\theta}, \quad (3.8)$$

as well as the angle variables

$$w^{r,\theta} = \Upsilon_{r,\theta} \lambda. \quad (3.9)$$

These angles allow us to take advantage of the separated nature of r and θ motion in Mino time: we treat r as a function only of w^r , θ as a function only of w^θ , and we treat w^r and w^θ as independent parameters. This allows us to Fourier decompose any function of the orbital worldline using standard action-angle variable techniques [11].

Before moving on, we should analyze the remaining coordinate motions of black hole orbits — the observer (Boyer-Lindquist) time t and the azimuthal angle ϕ . Both of these motions consist of a component that accumulates secularly as a function of λ , superposed on components which oscillate at Υ_r and Υ_θ . Let us analyze the oscillations first. From the geodesic equations (3.3) and (3.4), we know that $dt/d\lambda$ and $d\phi/d\lambda$ are functions only of r and θ . This means that they can be expanded in a Fourier series:

$$\frac{dt}{d\lambda} \equiv T(r, \theta) = \sum_{kn} T_{kn} e^{-i(k\Upsilon_\theta + n\Upsilon_r)\lambda}, \quad (3.10)$$

$$\frac{d\phi}{d\lambda} \equiv \Phi(r, \theta) = \sum_{kn} \Phi_{kn} e^{-i(k\Upsilon_\theta + n\Upsilon_r)\lambda}, \quad (3.11)$$

with the expansion coefficients given by

$$T_{kn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dw^r \int_0^{2\pi} dw^\theta T[r(w^r), \theta(w^\theta)] e^{i(kw^\theta + nw^r)}, \quad (3.12)$$

$$\Phi_{kn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dw^r \int_0^{2\pi} dw^\theta \Phi[r(w^r), \theta(w^\theta)] e^{i(kw^\theta + nw^r)}. \quad (3.13)$$

In these equations and in what follows, $r(w^r) \equiv r(\lambda = w^r/\Upsilon_r)$ and $\theta(w^\theta) \equiv \theta(\lambda = w^\theta/\Upsilon_\theta)$.

Because the functions $T(r, \theta)$ and $\Phi(r, \theta)$ are real, we have the following relations:

$$T_{-k, -n} = \bar{T}_{kn}, \quad (3.14)$$

$$\Phi_{-k, -n} = \bar{\Phi}_{kn}, \quad (3.15)$$

where the overbar denotes complex conjugation. The matrices T_{kn} and Φ_{kn} have another interesting property: T_{k0} and T_{0n} are non-zero, but $T_{kn} = 0$ if $k \neq 0$ and $n \neq 0$ (and likewise for Φ_{kn}). This lack of “crosstalk” between the θ and r harmonics is because $T(r, \theta)$ and $\Phi(r, \theta)$ have the form $f(r) + g(\theta)$. To take advantage of this property, we define

$$T_k^\theta \equiv T_{k0}, \quad T_n^r \equiv T_{0n}; \quad (3.16)$$

$$\Phi_k^\theta \equiv \Phi_{k0}, \quad \Phi_n^r \equiv \Phi_{0n}. \quad (3.17)$$

Using the complex conjugate relations and Eqs. (3.16) and (3.17), we rewrite the double sums appearing in the Fourier expansions (3.11) and (3.10) as a pair of single sums [18]:

$$\frac{dt}{d\lambda} \equiv T(r, \theta) = \Gamma + \sum_{k=1}^{\infty} (T_k^\theta e^{-ik\Upsilon_\theta \lambda} + \text{c.c.}) + \sum_{n=1}^{\infty} (T_n^r e^{-in\Upsilon_r \lambda} + \text{c.c.}); \quad (3.18)$$

$$\frac{d\phi}{d\lambda} \equiv \Phi(r, \theta) = \Upsilon_\phi + \sum_{k=1}^{\infty} (\Phi_k^\theta e^{-ik\Upsilon_\theta \lambda} + \text{c.c.}) + \sum_{n=1}^{\infty} (\Phi_n^r e^{-in\Upsilon_r \lambda} + \text{c.c.}). \quad (3.19)$$

The “c.c.” means the complex conjugate of the preceding term. We have pulled the $k = 0, n = 0$ terms out of these sums and defined

$$\Gamma = T_{00} , \quad (3.20)$$

$$\Upsilon_\phi = \Phi_{00} . \quad (3.21)$$

These numbers tell us about the secular, average rate at which ϕ and t accumulate with respect to λ .

Using these results, it is simple to integrate for $\phi(\lambda)$ and $t(\lambda)$:

$$t(\lambda) = \Gamma\lambda + \Delta t(\lambda) , \quad (3.22)$$

$$\phi(\lambda) = \Upsilon_\phi\lambda + \Delta\phi(\lambda) . \quad (3.23)$$

We have chosen $t(\lambda = 0) = 0 = \phi(\lambda = 0)$, and defined

$$\Delta t(\lambda) = \sum_{k=1}^{\infty} (\Delta t_k^\theta e^{-ik\Upsilon_\theta\lambda} + \text{c.c.}) + \sum_{n=1}^{\infty} (\Delta t_n^r e^{-in\Upsilon_r\lambda} + \text{c.c.}) ; \quad (3.24)$$

$$\Delta\phi(\lambda) = \sum_{k=1}^{\infty} (\Delta\phi_k^\theta e^{-ik\Upsilon_\theta\lambda} + \text{c.c.}) + \sum_{n=1}^{\infty} (\Delta\phi_n^r e^{-in\Upsilon_r\lambda} + \text{c.c.}) . \quad (3.25)$$

We have defined $\Delta t_j^{r,\theta} = iT_j^{r,\theta}/(j\Upsilon_{r,\theta})$ and $\Delta\phi_j^{r,\theta} = i\Phi_j^{r,\theta}/(j\Upsilon_{r,\theta})$. With this definition, we have separated $t(\lambda)$ and $\phi(\lambda)$ into pieces which accumulate secularly with λ plus pieces $\Delta t(\lambda)$ and $\Delta\phi(\lambda)$ that oscillate at harmonics of Υ_θ and Υ_r .

Since Ω_ϕ is the average rate at which ϕ accumulates as a function of t and since Γ and Υ_ϕ are the average rates at which t and ϕ accumulate as functions of λ ,

$$\Omega_\phi = \Upsilon_\phi/\Gamma . \quad (3.26)$$

The other frequencies are likewise related:

$$\Omega_\theta = \Upsilon_\theta/\Gamma , \quad (3.27)$$

$$\Omega_r = \Upsilon_r/\Gamma . \quad (3.28)$$

When performing a harmonic decomposition of any function, we will want to work in terms of the angles $w^j = \Upsilon_j\lambda$, for $j = r, \theta, \phi$, and the average accumulated time $\mathcal{T} = \Gamma\lambda$. In terms of these variables,

$$t(\mathcal{T}, w^\theta, w^r) = \mathcal{T} + \Delta t(w^\theta, w^r), \quad (3.29)$$

$$\phi(w^\phi, w^\theta, w^r) = w^\phi + \Delta\phi(w^\theta, w^r), \quad (3.30)$$

where

$$\Delta t(w^\theta, w^r) = \sum_{k=1}^{\infty} (\Delta t_k^\theta e^{-ikw^\theta} + \text{c.c.}) + \sum_{n=1}^{\infty} (\Delta t_n^r e^{-inw^r} + \text{c.c.}) ; , \quad (3.31)$$

$$\Delta\phi(w^\theta, w^r) = \sum_{k=1}^{\infty} (\Delta\phi_k^\theta e^{-ikw^\theta} + \text{c.c.}) + \sum_{n=1}^{\infty} (\Delta\phi_n^r e^{-inw^r} + \text{c.c.}) . \quad (3.32)$$

Putting all of this together, the Fourier expansion coefficients \tilde{f}_{kn} of any function of the form $f[r(\lambda), \theta(\lambda)]$ is

$$\tilde{f}_{kn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} dw^r \int_0^{2\pi} dw^\theta f[r(w^r), \theta(w^\theta)] e^{i(kw^\theta + nw^r)} , \quad (3.33)$$

It is useful to note that the worldline z^α can be reorganized in a similar form, separating the oscillations from the secular accumulations:

$$z^\alpha(\lambda) = z_{\text{sec}}^\alpha(\lambda) + \Delta z^\alpha[r(\lambda), \theta(\lambda)] , \quad (3.34)$$

where $z_{\text{sec}}^\alpha(\lambda) = (\Gamma\lambda, 0, 0, \Upsilon_\phi\lambda)$, and where

$$\Delta z^\alpha[r, \theta] = [\Delta t(r, \theta), r, \theta, \Delta\phi(r, \theta)] \quad (3.35)$$

can be expanded using the simple Fourier coefficients described by Eq. (3.33) with $f = z^\alpha$. This leaves the worldline in the desirable form

$$z^\alpha(\lambda) = z_{\text{sec}}^\alpha(\lambda) + \sum_{kn} \Delta z_{kn}^\alpha e^{-i(k\Upsilon_\theta + n\Upsilon_r)\lambda} . \quad (3.36)$$

By making use of Eq. (3.36), even rather complicated functional forms turn out to have a straightforward harmonic description.

IV. CONVERTING FOURIER EXPANSION COEFFICIENTS

There are two ways of Fourier expanding a function of the form $f[r(t), \theta(t)]$ which are essentially equivalent: we can expand in observer time t ,

$$f[r(t), \theta(t)] = \sum_{kn} f_{kn} e^{-i\Omega_{kn}t} ; \quad (4.1)$$

or, we can expand in Mino time λ ,

$$f[r(\lambda), \theta(\lambda)] = \sum_{kn} \tilde{f}_{kn} e^{-i\Upsilon_{kn}\lambda} . \quad (4.2)$$

We have defined

$$\begin{aligned} \Omega_{kn} &= k\Omega_\theta + n\Omega_r , \\ \Upsilon_{kn} &= k\Upsilon_\theta + n\Upsilon_r . \end{aligned} \quad (4.3)$$

From the standpoint of measurable physics, the expansion (4.1) is more interesting — the components f_{kn} tell us about the harmonic structure of f as seen by distant observers. However, the expansion (4.2) is far more accessible — using Eq. (3.33), it is straightforward to compute the expansion components \tilde{f}_{kn} . In this section, we show how to convert the accessible components \tilde{f}_{kn} into the measurable components f_{kn} .

We begin by taking the Fourier transform of $f[r(t), \theta(t)]$. Using Eq. (4.1), we have

$$\begin{aligned} \sum_{kn} f_{kn} \delta(\omega - \Omega_{kn}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dt f[r(t), \theta(t)] e^{i\omega t} , \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \frac{dt}{d\lambda} f[r(\lambda), \theta(\lambda)] e^{i\omega t(\lambda)} . \end{aligned} \quad (4.4)$$

Our goal is to evaluate the integral on the right-hand side of (4.4) and to find an expression relating f_{kn} to \tilde{f}_{kn} . To do so, we take advantage of the Fourier expansion for $t(\lambda)$ previously established, (3.22).

We now insert $dt/d\lambda = T(r, \theta)$ and $e^{i\omega t(\lambda)} = e^{i\omega\Gamma\lambda} \times e^{i\omega\Delta t(\lambda)}$, under the integral:

$$\begin{aligned} \sum_{kn} f_{kn} \delta(\omega - \Omega_{kn}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \left\{ T[r(\lambda), \theta(\lambda)] f[r(\lambda), \theta(\lambda)] e^{i\omega\Delta t(\lambda)} \right\} e^{i\omega\Gamma\lambda} , \\ &\equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \mathcal{F}[r(\lambda), \theta(\lambda), \omega] e^{i\omega\Gamma\lambda} . \end{aligned} \quad (4.5)$$

We next wish to insert the Fourier expansion of $\mathcal{F}[r(\lambda), \theta(\lambda), \omega]$ under the integral. We first write this function in terms of the angle variables:

$$\mathcal{F}(w^\theta, w^r, \omega) = T[r(w^r), \theta(w^\theta)] e^{i\omega\Delta t(w^\theta, w^r)} f[r(w^r), \theta(w^\theta)] . \quad (4.6)$$

The expansion of $\mathcal{F}[r(\lambda), \theta(\lambda), \omega]$ is

$$\mathcal{F}[r(\lambda), \theta(\lambda), \omega] = \sum_{ab} \mathcal{F}_{ab}(\omega) e^{-i\Upsilon_{ab}\lambda} , \quad (4.7)$$

where

$$\mathcal{F}_{ab}(\omega) = \frac{1}{(2\pi)^2} \int_0^{2\pi} dw^r \int_0^{2\pi} dw^\theta \mathcal{F}(w^\theta, w^r, \omega) e^{iaw^\theta} e^{ibw^r}. \quad (4.8)$$

Inserting the expansion (4.6) into Eq. (4.5), we find

$$\begin{aligned} \sum_{kn} f_{kn} \delta(\omega - \Omega_{kn}) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \sum_{ab} \mathcal{F}_{ab}(\omega) e^{i(\omega\Gamma - \Upsilon_{ab})\lambda}, \\ &= \sum_{ab} \mathcal{F}_{ab}(\omega) \delta(\omega\Gamma - \Upsilon_{ab}), \\ &= \Gamma^{-1} \sum_{ab} \mathcal{F}_{ab}(\omega) \delta(\omega - \Omega_{ab}), \\ &= \Gamma^{-1} \sum_{ab} \mathcal{F}_{ab}(\Omega_{ab}) \delta(\omega - \Omega_{ab}). \end{aligned} \quad (4.9)$$

Equating the left-hand sides and right-hand sides, we read off

$$f_{kn} = \mathcal{F}_{kn}(\Omega_{kn})/\Gamma. \quad (4.10)$$

We use this form of the Fourier expansion coefficients in all of our calculations.

V. EXAMPLES

In this section we use the methods described above in two test cases. We first decompose and reconstruct a simple function of the form $f(r, \theta)$. We then show how to decompose a more complicated function which appears in black hole perturbation calculations.

A. Simple test case

We now test our prescription by computing the expansion coefficients of a test function and showing that the reconstructed time series agrees with the original function. Our test function is $\zeta \equiv r \cos \theta$, where r and θ are the Boyer-Lindquist coordinates. Our calculations are actually performed using the tips given in Appendix A. In particular, we map the radial coordinate r to a coordinate ψ defined via

$$r = \frac{pM}{1 + \varepsilon \cos \psi}, \quad (5.1)$$

a reparameterization commonly used in Newtonian orbital dynamics [20]. Whereas r oscillates from periapsis [$r_{\text{peri}} = pM/(1 + \varepsilon)$] to apoapsis [$r_{\text{ap}} = pM/(1 - \varepsilon)$] and back, ψ winds secularly from 0 (periapsis) to π (apoapsis) and beyond. We truncate all infinite sums at some finite value N , discussed below. We also map the θ motion to a coordinate χ via

$$\cos \theta = \sqrt{z_-(a, E, \iota)} \cos \chi, \quad (5.2)$$

where $z_-(a, E, \iota)$ is one root of a quadratic equation defined in Appendix A. The inclination angle ι relates the angular momentum L_z to the Carter constant Q :

$$\cos \iota = \frac{L_z}{\sqrt{L_z^2 + Q}}. \quad (5.3)$$

Although ι is not quite the geometrical angle describing an orbit's excursion from the equatorial plane, it is closely related, and has other convenient properties (see, e.g., Ref. [21] for further discussion).

It's worth noting that the parameterization (5.1) makes manifestly clear that the radial motion has a slowly converging Fourier expansion for large eccentricity. Using the binomial expansion,

$$r = pM \sum_{n=0}^{\infty} (-1)^n \varepsilon^n \cos^n \psi. \quad (5.4)$$

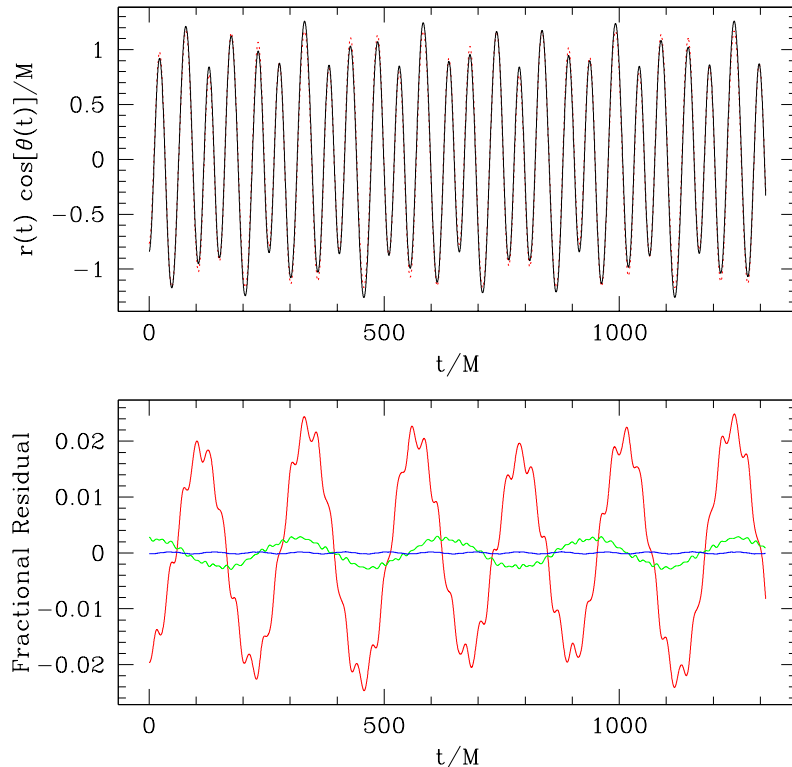


FIG. 1: The function $\zeta(t) = r(t) \cos[\theta(t)]$ for orbits about a black hole with $a = 0.9M$, and with parameters $p = 3$, $\varepsilon = 0.2$, $\iota = 20^\circ$. Top panel shows this function computed directly from the geodesic equations (solid black line) as well as the time series constructed using Eq. (4.10) (dotted line; red in color version). In the reconstructed time series $\zeta_{\text{rec}}(t)$, we have truncated the infinite sums at $N = 1$. We find remarkably good agreement despite the small number of terms kept in the sum. The bottom panel shows the fractional residual, $[\zeta(t) - \zeta_{\text{rec}}(t)]/\zeta_{\text{max}}$, for several values of N . The largest differences (red in color version) are for $N = 1$, and have a magnitude of about 0.025. The next largest (green in color version) are for $N = 2$ and have a magnitude of about 0.003. The smallest differences (blue in color version) are for $N = 3$ and have a magnitude of about 0.0002.

The amplitude of radial harmonic n is roughly ε smaller than the amplitude of harmonic $n - 1$. When we truncate our sums at some finite value N , we expect that a reconstructed time series will have a fractional error of order ε^{N+1} (the amplitude of the next neglected coefficient). Hence, many harmonics will be needed as ε approaches 1. This slow convergence has been noted in studies of gravitational radiation reaction on eccentric black hole orbits [22, 23].

The top panel in Figs. 1 and 2 show the function $\zeta(t)$ computed in two different ways. The solid black line shows $\zeta(t)$ constructed by direct integration of the geodesic equations; the dotted line (red in the color version) shows $\zeta(t)$ reconstructed from a Fourier expansion using Eq. (4.10). The lower panel of these figures shows the fractional residual, $[\zeta(t) - \zeta_{\text{rec}}(t)]/\zeta_{\text{max}}$, where $\zeta_{\text{rec}}(t)$ is the reconstructed timeseries and $\zeta_{\text{max}} = r_{\text{max}} \cos \theta_{\text{min}}$.

Figure 1 compares $\zeta(t)$ and $\zeta_{\text{rec}}(t)$ for an orbit with $p = 3$, $\varepsilon = 0.3$, $\iota = 20^\circ$; the black hole's spin parameter $a = 0.9M$. The reconstructed timeseries converges to $\zeta(t)$ rather quickly: even when the infinite sums are truncated at $N = 1$, the maximum deviation is only a few percent; the difference between $\zeta(t)$ and $\zeta_{\text{rec}}(t)$ for $N = 1$ is barely discernible on the plot. The convergence is quite a bit slower when $\varepsilon = 0.6$. The motion shown in Fig. 2 is for an orbit with $p = 4$, $\varepsilon = 0.6$, and $\iota = 50^\circ$. The timeseries is within 2% of $\zeta(t)$ when $N = 4$; reducing the error by a further factor of ten requires increasing N to 8. It's worth noting that, even when the amplitude error is relatively large, the phase alignment of $\zeta_{\text{rec}}(t)$ with $\zeta(t)$ appears to be very good. This bodes well for problems that rely on an accurate phase match between data and a model (or template).

Notice, in both Figs. (1) and (2), that the differences between $\zeta(t)$ and the reconstructed time series are *not* precisely periodic: the wiggles in the lower panels of these figures do not repeat themselves regularly. This is a manifestation of the quasi-periodic nature of the orbital motion. As more terms are kept in these sums, we are more successful at capturing this quasi-periodic motion, and the magnitude of these wiggles quickly becomes small.

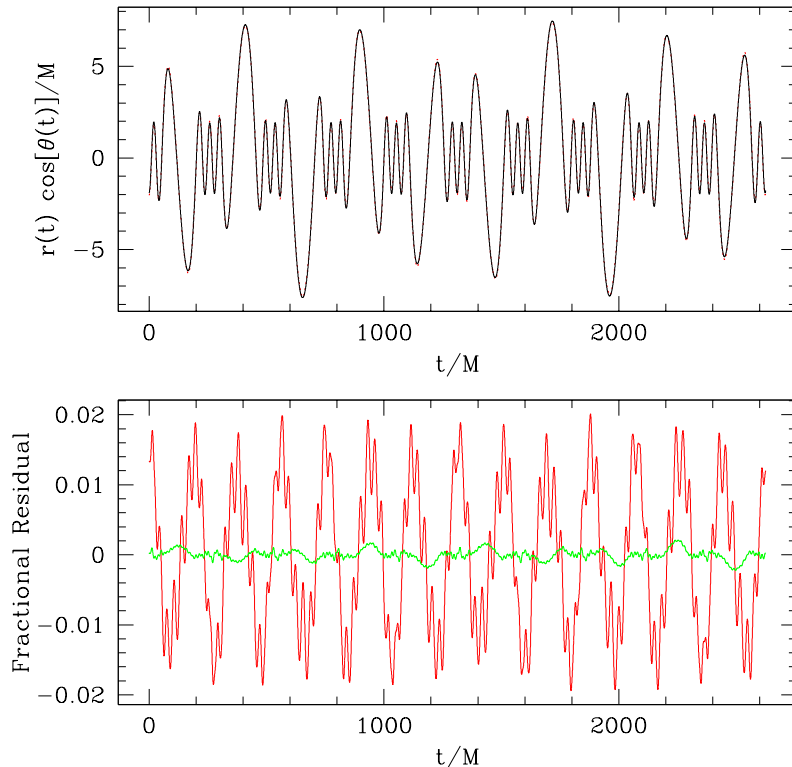


FIG. 2: The function $\zeta(t)$ for orbits about a black hole with $a = 0.9M$ and with parameters $p = 4$, $\varepsilon = 0.6$, $\iota = 50^\circ$. Top panel shows this function computed directly from the geodesic equations (solid black line) plus the time series constructed using Eq. (4.10) (dotted line; red in color version). We have truncated sums in the time series at $N = 4$. We begin to see the need to keep a large number of terms at this relatively large eccentricity. The bottom panel shows the fractional residual for several values of N . The largest differences (red in color version) are for $N = 4$, and have a magnitude of about 0.02. The smaller differences (green in color version) are for $N = 8$ and have a magnitude of about 0.002.

B. Black hole perturbations

We now apply these techniques to a problem taken from black hole perturbation theory [10]. Our goal is to understand how to decompose a complex function ψ_4 which describes how a small body perturbs the spacetime curvature of a Kerr black hole. From ψ_4 , one can extract information about gravitational-wave emission and radiative backreaction on small compact objects orbiting massive black holes — extreme mass ratio binaries. When frequency domain perturbation theory is used to study this problem, ψ_4 is expanded in multipoles and in a harmonic series of the fundamental orbital frequencies. The gravitational waves generated by the system and their backreaction onto the orbit can then be extracted from that harmonic/multipolar expansion. Aside from being of great interest to the current authors, this example nicely illustrates the principles of this Fourier decomposition for functionals more complicated than the previous simple example. We will not dwell too much on the mathematics of black hole perturbation theory, but will point the reader to references where appropriate.

The function ψ_4 can be decomposed into multipoles as [10]

$$\psi_4(t_f, r_f, \theta_f, \phi_f) = \rho^{-4} \sum_{lm} \int d\omega R_{lm}(r_f, \omega) S_{lm}(\theta_f, a\omega) e^{i(m\phi_f - \omega t_f)}, \quad (5.5)$$

where the angular function S is a spin weighted spheroidal harmonic, and the radial function R is a solution of a second order ordinary differential equation known as the Teukolsky equation (see Refs. [10, 22, 23, 24, 25, 26, 27] for a detailed discussion of the Teukolsky equation). The subscript f on the coordinates is a reminder that $(t_f, r_f, \theta_f, \phi_f)$ denotes a field point. Coordinates without the subscript will refer to the location of a body orbiting the black hole. We will now rewrite ψ_4 completely in terms of sums, eliminating the need for the integral over ω .

The radial function can be written in the form [22, 23, 24, 25, 26]

$$R_{lm}(r_f, \omega) = Z_{lm}^H(r_f, \omega) R_{lm}^\infty(r_f, \omega) + Z_{lm}^\infty(r_f, \omega) R_{lm}^H(r_f, \omega) , \quad (5.6)$$

where $R_{lm}^{H,\infty}(r_f, \omega)$ are the two independent solutions to the source-free Teukolsky equation and where the functions Z are

$$Z_{lm}^\star(r_f, \omega) = \int dt e^{i[\omega t - m\phi(t)]} I_{lm}^\star[r(t), \theta(t), r_f, \omega] \quad (5.7)$$

for $\star = H, \infty$. The function $I_{lm}^\star[r(t), \theta(t), r_f, \omega]$ depends upon the orbital worldline of the body perturbing the black hole spacetime. See Ref. [23] for discussion in the case of a body in an equatorial, eccentric orbit; see Ref. [26] for the case of a body in an orbit that is inclined but of constant radius. (The general case, for orbits that are inclined and eccentric, is in preparation [27].)

We next rewrite Eq. (5.7) as an integral over λ

$$Z_{lm}^\star(r_f, \omega) = \int d\lambda e^{i[\omega t(\lambda) - m\phi(\lambda)]} \mathcal{I}_{lm}^\star[r(\lambda), \theta(\lambda), r_f, \omega] , \quad (5.8)$$

where $\mathcal{I}_{lm}^\star = I_{lm}^\star dt/d\lambda$. Now we insert Eqs. (3.22) and (3.23) into (5.8) so that we have

$$Z_{lm}^\star(r_f, \omega) = \int d\lambda e^{i[\omega \Gamma \lambda - m \Upsilon_\phi \lambda]} J_{lm}^\star[r(\lambda), \theta(\lambda), r_f, \omega] , \quad (5.9)$$

where

$$J_{lm}^\star[r(\lambda), \theta(\lambda), r_f, \omega] = \mathcal{I}_{lm}^\star[r(\lambda), \theta(\lambda), r_f, \omega] e^{i\{\omega \Delta t[r(\lambda), \theta(\lambda)] - m \Delta \phi[r(\lambda), \theta(\lambda)]\}} , \quad (5.10)$$

with Δt and $\Delta \phi$ given by Eqs. (3.24) and (3.25). Since J_{lm}^\star depends on λ only through $r(\lambda)$ and $\theta(\lambda)$, it can be expanded as

$$J_{lm}^\star[r(\lambda), \theta(\lambda), r_f, \omega] = \sum_{kn} J_{lmkn}^\star(r_f, \omega) e^{-i \Upsilon_{kn} \lambda} . \quad (5.11)$$

Putting this into Eq. (5.9) and performing the integral gives

$$\begin{aligned} Z_{lm}^\star(r_f, \omega) &= 2\pi \sum_{jk} J_{lmkn}^\star(r_f, \omega) \delta(\omega \Gamma - \Upsilon_{mkn}) \\ &= \frac{2\pi}{\Gamma} \sum_{jk} J_{lmkn}^\star(r_f, \omega) \delta(\omega - \Omega_{mkn}) , \end{aligned} \quad (5.12)$$

where

$$\Omega_{mkn} = \Upsilon_{mkn}/\Gamma = m\Omega_\phi + k\Omega_\theta + n\Omega_r . \quad (5.13)$$

Finally, when we substitute Eq. (5.12) into Eq. (5.5) we obtain

$$\psi_4(t_f, r_f, \theta_f, \phi_f) = \frac{1}{\rho^4} \sum_{lmkn} R_{lmkn}(r_f) S_{lmkn}(\theta_f) e^{i(m\phi_f - \Omega_{mkn} t_f)} , \quad (5.14)$$

where

$$\begin{aligned} S_{lmkn}(\theta_f) &= S_{lm}(\theta_f, a\Omega_{mkn}) \\ R_{lmkn}(r_f) &= Z_{lmkn}^H(r_f) R_{lm}^\infty(r_f, \Omega_{mkn}) + Z_{lmkn}^\infty(r_f) R_{lm}^H(r_f, \Omega_{mkn}) \\ Z_{lmkn}^\star(r_f) &= 2\pi J_{lmkn}^\star(r_f, \Omega_{mkn})/\Gamma . \end{aligned} \quad (5.15)$$

VI. CONCLUSION

With the techniques described in this paper, it should now be a relatively simple matter to describe functions of Kerr black hole orbits in the frequency domain. Although the motion is not truly periodic with respect to observer time t , it *is* periodic with respect to Mino time λ . It is thus quite simple to represent functions using frequencies Υ conjugate to Mino time. By using the fact that observer time t is itself periodic with respect to Mino time (after subtracting the secularly growing contribution), it is straightforward to convert the λ -Fourier expansion into a t -Fourier expansion.

As discussed in the Introduction, these techniques could find useful application to a variety of astrophysical problems involving Kerr black holes. One that is of particular interest to us is the problem of describing gravitational-wave emission from extreme mass ratio binaries [22, 23, 24, 25, 26]. Such systems are expected to be observable for future space based gravitational wave detectors. It should now be fairly straightforward to extend current black hole perturbation theory codes to handle the very interesting case of generic orbits — binaries in which the small body has both inclination with respect to the equatorial plane and non-zero eccentricity [27]. If we had been unable to exploit the discrete harmonic structure of these systems, such a generalization would have had an enormous computational cost. Combining that analysis with a scheme to compute the evolution of the Carter constant (using a rigorous computation of a self force [13], or perhaps using a cruder approximation [28]), it should then be possible to construct, in the adiabatic limit, the inspiral worldlines and waveforms followed by bodies spiraling into massive black holes.

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APPENDIX A: PRACTICAL EVALUATION OF THE KERR GEODESICS

In this appendix, we present some useful tools for handling functions of Kerr geodesics. A difficulty often encountered in evaluating these orbital motions is due to the presence of turning points in the motion: as the radial motion approaches periapsis and apoapsis, dr/d “time” passes through zero and switches sign (regardless of which time variable one uses). As the derivative approaches zero, one typically finds in a numerical evaluation that small stepsizes are needed to resolve the changing derivative; precision can be badly degraded in this case. The θ behavior exhibits similar behavior due to the turning points at θ_{\min} and $\pi - \theta_{\min}$.

A simple way to solve this behavior is to work with a functional form that automatically builds in the correct behavior as the turning points are approached. We first describe the transformation used to describe the θ motion. The core idea of this transformation has been known for quite some time [29], and has been used extensively in work on circular Kerr black hole orbits [26]; it turns out to be particularly simple to use when studying geodesics parameterized by Mino time. We then show a simple transformation that greatly simplifies the description of the radial motion. This transformation has also been used quite a bit in previous work [22, 23], but is worth discussing in the context of the Mino-time parameterization.

1. Motion in θ

We begin transforming the θ motion by first defining the variable $z = \cos^2 \theta$. Equation (3.2) becomes

$$\begin{aligned} \frac{d\theta}{d\lambda} &= \pm \sqrt{\frac{z^2 [a^2(1 - E^2)] - z [Q + L_z^2 + a^2(1 - E^2)] + Q}{1 - z}} \\ &= \pm \sqrt{\frac{\beta(z_+ - z)(z_- - z)}{1 - z}}. \end{aligned} \tag{A1}$$

The plus sign corresponds to motion from θ_{\min} to $\pi - \theta_{\min}$, and vice versa for the minus sign. We have defined $\beta = a^2(1 - E^2)$; z_{\pm} are the two roots of the quadratic in the top line of Eq. (A1).

We next define the variable χ : $z = z_- \cos^2 \chi$. As χ varies from 0 to 2π , θ oscillates through its full range of motion, from θ_{\min} to $\pi - \theta_{\min}$ and back. Examining $dz/d\theta$ and $dz/d\chi$ we see that

$$\begin{aligned} \frac{d\chi}{d\theta} &= \sqrt{\frac{1-z}{z_- - z}}, & 0 \leq \chi \leq \pi; \\ &= -\sqrt{\frac{1-z}{z_- - z}}, & \pi \leq \chi \leq 2\pi. \end{aligned} \quad (\text{A2})$$

Combining Eqs. (A1) and (A2), we obtain the geodesic equation for χ :

$$\begin{aligned} \frac{d\chi}{d\lambda} &= \sqrt{\beta(z_+ - z)} \\ &= \sqrt{\beta(z_+ - z_- \cos^2 \chi)}. \end{aligned} \quad (\text{A3})$$

Using Eq. (A3), it is straightforward to find λ for all χ . First, define

$$\lambda_0(\chi) = \frac{1}{\sqrt{\beta z_+}} \left[K(\sqrt{z_-/z_+}) - F(\pi/2 - \chi, \sqrt{z_-/z_+}) \right]; \quad (\text{A4})$$

note that

$$\lambda_0(\pi/2) = \frac{1}{\sqrt{\beta z_+}} K(\sqrt{z_-/z_+}). \quad (\text{A5})$$

In these equations, the function $F(\varphi, k)$ is the incomplete elliptic integral of the first kind, and $K(k)$ is the complete elliptic integral of the first kind (using the notation of [30]). Then,

$$\begin{aligned} \lambda(\chi) &= \lambda_0(\chi) & 0 \leq \chi \leq \pi/2 \\ &= \frac{2}{\sqrt{\beta z_+}} K(\sqrt{z_-/z_+}) - \lambda_0(\pi - \chi) & \pi/2 \leq \chi \leq \pi \\ &= \frac{2}{\sqrt{\beta z_+}} K(\sqrt{z_-/z_+}) + \lambda_0(\chi - \pi) & \pi \leq \chi \leq 3\pi/2 \\ &= \frac{4}{\sqrt{\beta z_+}} K(\sqrt{z_-/z_+}) - \lambda_0(2\pi - \chi) & 3\pi/2 \leq \chi \leq 2\pi; \end{aligned} \quad (\text{A6})$$

also

$$\Lambda_\theta = \frac{4}{\sqrt{\beta z_+}} K(\sqrt{z_-/z_+}). \quad (\text{A7})$$

This form of Λ_θ is perfectly well behaved even for orbits that are confined to the equatorial plane ($\theta_{\min} = \pi/2$); this is not the case for the original form (3.7).

By combining Eqs. (3.8), (A3), and (A7) it is trivial to change variables so that integrals of w^θ become integrals over χ :

$$w^\theta(\chi) = \Upsilon_\theta \lambda(\chi); \quad (\text{A8})$$

$$\begin{aligned} \frac{dw^\theta}{d\chi} &= \Upsilon_\theta \frac{d\lambda}{d\chi} \\ &= \frac{2\pi}{\Lambda_\theta} \frac{1}{\sqrt{\beta(z_+ - z_- \cos^2 \chi)}} \\ &= \frac{\pi}{2K(\sqrt{z_-/z_+})} \frac{1}{\sqrt{1 - (z_-/z_+) \cos^2 \chi}}. \end{aligned} \quad (\text{A9})$$

Equations (A8) and (A9) are used in our applications to perform all integrals with respect to the angle variable w^θ .

2. Motion in r

We use a similar trick to simplify the radial motion. First, we reparameterize the instantaneous orbital radius as

$$r = \frac{pM}{1 + \varepsilon \cos \psi} . \quad (\text{A10})$$

Such a reparameterization is commonly used to study Keplerian orbits in Newtonian theory [20]; though relativistic orbits are not closed ellipses, the form (A10) remains very useful. The parameter ε can thus be interpreted as the eccentricity, ψ as the orbital anomaly, and p as the semi-latus rectum. As ψ varies from 0 to π , r varies from periapsis (closest approach) to apoapsis (furthest distance):

$$r_{\text{peri}} = \frac{pM}{1 + \varepsilon} , \quad (\text{A11})$$

$$r_{\text{ap}} = \frac{pM}{1 - \varepsilon} . \quad (\text{A12})$$

To proceed, we must do some massaging of the function $R(r)$ defined in Eq. (2.1). It is a quartic function of r , and thus has 4 roots:

$$\begin{aligned} R(r) &= (E^2 - 1)r^4 + 2Mr^3 + [a^2(E^2 - 1) - L_z^2 - Q]r^2 + 2M[Q + (aE - L_z)^2]r - a^2Q \\ &= (1 - E^2)(r_1 - r)(r - r_2)(r - r_3)(r - r_4) . \end{aligned} \quad (\text{A13})$$

The second line of Eq. (A13) is written in a way that is manifestly positive for bound orbits ($E < 1$). The roots are ordered such that $r_1 \geq r_2 \geq r_3 \geq r_4$; bound motion occurs for $r_1 \geq r \geq r_2$. From these definitions, it is clear that $r_1 \equiv r_{\text{ap}}$, and $r_2 \equiv r_{\text{peri}}$.

The radii r_3 and r_4 do not correspond to turning points of the small body's motion, but of course still represent zeros of the function R . (In fact, r_4 is typically inside the event horizon; when $Q = 0$ or $a = 0$, $r_4 = 0$.) It turns out to be useful to remap these radii as follows:

$$r_3 = \frac{p_3 M}{1 - \varepsilon} , \quad (\text{A14})$$

$$r_4 = \frac{p_4 M}{1 + \varepsilon} . \quad (\text{A15})$$

This remapping is simply for mathematical convenience; the parameters p_3 and p_4 have no particular physical meaning.

It is now a simple matter to derive the geodesic equation for ψ :

$$\begin{aligned} \frac{d\psi}{d\lambda} &= \frac{M\sqrt{1 - E^2} [(p - p_3) - \varepsilon(p + p_3 \cos \psi)]^{1/2} [(p - p_4) + \varepsilon(p - p_4 \cos \psi)]^{1/2}}{1 - \varepsilon^2} \\ &\equiv P(\psi); . \end{aligned} \quad (\text{A16})$$

As with the χ reparameterization of the θ motion, it is straightforward to find $\lambda(\psi)$ using Eq. (A16):

$$\lambda(\psi) = \int_0^\psi \frac{d\psi'}{P(\psi')} . \quad (\text{A17})$$

In our applications, we evaluate this integral numerically. It is possible that an analytic form could be found in terms of elliptic integrals (though it appears to require more algebraic fortitude than these authors could muster). In any practical application, it is unlikely that such a form will be more useful or accurate than a numerical evaluation of (A17).

Note in particular that

$$\Lambda_r = \int_0^{2\pi} \frac{d\psi'}{P(\psi')} . \quad (\text{A18})$$

This form of Λ_r is well-behaved in the limit of circular orbits.

Finally, we use these results to convert integrals over w^r into integrals over ψ : combining Eqs. (3.8), (A16), and (A18), we have

$$w^r(\psi) = \Upsilon_r \lambda(\psi); \quad (\text{A19})$$

$$\begin{aligned} \frac{dw^r}{d\psi} &= \Upsilon_r \frac{d\lambda}{d\psi} \\ &= \frac{2\pi}{\Lambda_r} \frac{1}{P(\psi)}. \end{aligned} \quad (\text{A20})$$

We use Eqs. (A19) and (A20) to perform all integrals with respect to w^r .

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